Modal Expansion of Dyadic Green's Functions of the Cylindrical Chirowaveguide

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Abstract— **The dyadic Green's functions of the cylindrical chirowaveguide are derived by modal expansion. Bohren's decomposition of the electromagnetic field is used to obtain the vector wave functions. The magnetic dyadic Green's function that is purely solenoidal is first derived and the electric dyadic Green's function is then obtained by manipulating the magnetic dyadic Green's function. The singular term in the expression of the electric dyadic Green's function is reinstated through the manipulation procedure.**

I. Introduction

Recently, the theory of chirowaveguide has attracted much research interest because of the potential applications of chirowaveguides in the area of electromagnetics. Dyadic Green's functions in an unbounded chiral medium [1], [2] have been found. One- and two-dimensional dyadic Green's functions in chiral media have also been determined [3]. Although Engheta *et al.* [4] obtained a modal expansion of the electric dyadic Green's function in terms of the spherical vector wave functions for the case of scattering from a chiral sphere, their result is not a complete expansion [5] and is only applicable to source-free regions. More recently, Li *et al.* [6] formulated the dyadic Green's functions for a radially multilayered chiral sphere and the singular term accounting for the electric field in the source point was reinstated, but the reason for this was not explained. In this Letter, we seek the modal expansion of the dyadic Green's functions of the cylindrical chirowaveguide that have not yet been available in the literature. The magnetic dyadic Green's function, which is purely solenoidal, is derived first, and the electric dyadic Green function is then obtained through a manipulation of the magnetic dyadic Green's function. It is found that the singular term of the electric dyadic Green's function is just a natural outcome of the manipulation procedure.

II. Formulation

Consider a lossless, reciprocal, homogeneous chiral medium. The electromagnetic field is characterized by the following constitutive equations [7]–[9]:

$$\boldsymbol{D} = \epsilon[\boldsymbol{E} + \beta\nabla \times \boldsymbol{E}] \tag{1a}$$

$$\boldsymbol{B} = \mu[\boldsymbol{H} + \beta\nabla \times \boldsymbol{H}] \tag{1b}$$

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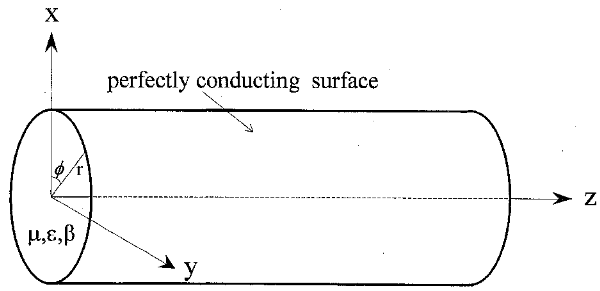

Fig. 1. The geometry of the cylindrical chirowaveguide.

where ϵ and μ are, respectively, the permittivity and permeability of the medium and β is a measure of the chirality. In the following formulation, the electromagnetic field, $(\boldsymbol{E}, \boldsymbol{H})$ is assumed to be time-harmonic with $e^{-j\omega t}$ dependence. Putting (1a) and (1b) into Maxwell's equations, we obtain the vector wave equations

$$\begin{aligned}\nabla \times \nabla \times \boldsymbol{E}(\boldsymbol{R}) - 2\gamma^2\beta\nabla \times \boldsymbol{E}(\boldsymbol{R}) - \gamma^2\boldsymbol{E}(\boldsymbol{R}) \\ = j\omega\mu(\gamma/k)^2(1 + \beta\nabla\times)\boldsymbol{J}(\boldsymbol{R})\end{aligned} \tag{2a}$$

$$\begin{aligned}\nabla \times \nabla \times \boldsymbol{H}(\boldsymbol{R}) - 2\gamma^2\beta\nabla \times \boldsymbol{H}(\boldsymbol{R}) - \gamma^2\boldsymbol{H}(\boldsymbol{R}) \\ = (\gamma/k)^2\nabla \times \boldsymbol{J}(\boldsymbol{R})\end{aligned} \tag{2b}$$

where $\gamma^2 = k^2/(1-k^2\beta^2), k = \omega\sqrt{\mu\epsilon}$, and $\boldsymbol{J}$ is the impressed current source density. The equations governing the electric dyadic Green's function, $\overline{\overline{\boldsymbol{G}}}_{\boldsymbol{e}}$, and the magnetic dyadic Green's function, $\overline{\overline{\boldsymbol{G}}}_{\boldsymbol{m}}$, are therefore given by

$$\begin{aligned}\nabla \times \nabla \times \overline{\overline{\boldsymbol{G}}}_{\boldsymbol{e}}(\boldsymbol{R}, \boldsymbol{R}') - 2\gamma^2\beta\nabla \\ \times \overline{\overline{\boldsymbol{G}}}_{\boldsymbol{e}}(\boldsymbol{R}, \boldsymbol{R}') - \gamma^2\overline{\overline{G}}_e(\boldsymbol{R}, \boldsymbol{R}') \\ = j\omega\mu(\gamma/k)^2(1 + \beta\nabla\times)\overline{\overline{\boldsymbol{I}}}\delta(\boldsymbol{R} - \boldsymbol{R}')\end{aligned} \tag{3a}$$

$$\begin{aligned}\nabla \times \nabla \times \overline{\overline{\boldsymbol{G}}}_{\boldsymbol{m}}(\boldsymbol{R}, \boldsymbol{R}') - 2\gamma^2\beta\nabla \\ \times \overline{\overline{\boldsymbol{G}}}_{\boldsymbol{m}}(\boldsymbol{R}, \boldsymbol{R}') - \gamma^2\overline{\overline{\boldsymbol{G}}}_{\boldsymbol{m}}(\boldsymbol{R}, \boldsymbol{R}') \\ = (\gamma/k)^2\nabla \times \overline{\overline{\boldsymbol{I}}}\delta(\boldsymbol{R} - \boldsymbol{R}')\end{aligned} \tag{3b}$$

where $\overline{\overline{\boldsymbol{I}}}$ is the unit dyad and $\delta(\boldsymbol{R}-\boldsymbol{R}')$ is the three-dimensional delta function. The boundary conditions of a chirowaveguide with perfectly conducting walls are

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0} \tag{4a}$$

$$\boldsymbol{n} \times \overline{\overline{\boldsymbol{G}}}_{\boldsymbol{e}}(\boldsymbol{R}, \boldsymbol{R}') = \boldsymbol{0} \tag{4b}$$

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where $\boldsymbol{n}$ is an outward-pointing unit normal vector defined on the surface of the waveguide. By using Bohren's decomposition of the electromagnetic field [10], we may transform the electromagnetic field as follows:

$$\begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} t & 1 \\ 1 & -1/t \end{bmatrix} \begin{bmatrix} \boldsymbol{Q_1} \\ \boldsymbol{Q_2} \end{bmatrix} \tag{5}$$

where

$$t = j\sqrt{\mu/\epsilon} \tag{6}$$

and when the transformed fields, $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$, satisfy the following two equations:

$$\nabla \times \begin{bmatrix} \boldsymbol{Q_1} \\ \boldsymbol{Q_2} \end{bmatrix} = \begin{bmatrix} k_+ & 0 \\ 0 & -k_- \end{bmatrix} \begin{bmatrix} \boldsymbol{Q_1} \\ \boldsymbol{Q_2} \end{bmatrix} \tag{7}$$

$$\nabla \times \nabla \times \begin{bmatrix} \boldsymbol{Q_1} \\ \boldsymbol{Q_2} \end{bmatrix} = \begin{bmatrix} k_+^2 & 0 \\ 0 & k_-^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{Q_1} \\ \boldsymbol{Q_2} \end{bmatrix} \tag{8}$$

with

$$k_+ = k/(1 - k\beta) \tag{9a}$$

$$k_- = k/(1 + k\beta). \tag{9b}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ then satisfy, respectively, (2a) and (2b). In the case of the cylindrical chirowaveguide as shown in Fig. 1, the following forms of $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$ defined in the cylindrical coordinates obviously satisfy (7) and (8):

$$\boldsymbol{Q_{1\lambda_1 n}}(h) = A_{\lambda_1 n}[\boldsymbol{M_{1\lambda_1 n}}(h) + \boldsymbol{N_{1\lambda_1 n}}(h)] \tag{10a}$$

$$\boldsymbol{Q_{2\lambda_2 n}}(h) = B_{\lambda_2 n}[\boldsymbol{M_{2\lambda_2 n}}(h) + \boldsymbol{N_{2\lambda_2 n}}(h)] \tag{10b}$$

where

$$\boldsymbol{M_{1\lambda_1 n}}(h) = \begin{bmatrix} jn\,\dfrac{J_n(\lambda_1 r)}{r} \\ -\dfrac{\partial J_n(\lambda_1 r)}{\partial r} \\ 0 \end{bmatrix} e^{jn\phi} e^{jhz} = \frac{1}{k_+} \nabla \times \boldsymbol{N_{1\lambda_1 n}}(h) \tag{11a}$$

$$\boldsymbol{N_{1\lambda_1 n}}(h) = \frac{1}{k_+} \begin{bmatrix} jh\,\dfrac{\partial J_n(\lambda_1 r)}{\partial r} \\ -nh\,\dfrac{J_n(\lambda_1 r)}{r} \\ \lambda_1^2 J_n(\lambda_1 r) \end{bmatrix} e^{jn\phi} e^{jhz} = \frac{1}{k_+} \nabla \times \boldsymbol{M_{1\lambda_1 n}}(h) \tag{11b}$$

$$\boldsymbol{M_{2\lambda_2 n}}(h) = \begin{bmatrix} jn\,\dfrac{J_n(\lambda_2 r)}{r} \\ -\dfrac{\partial J_n(\lambda_2 r)}{\partial r} \\ 0 \end{bmatrix} e^{jn\phi} e^{jhz} = \frac{1}{k_-} \nabla \times \boldsymbol{N_{2\lambda_2 n}}(h) \tag{11c}$$

$$\boldsymbol{N_{2\lambda_2 n}}(h) = -\frac{1}{k_-} \begin{bmatrix} jh\,\dfrac{\partial J_n(\lambda_2 r)}{\partial r} \\ -nh\,\dfrac{J_n(\lambda_2 r)}{r} \\ \lambda_2^2 J_n(\lambda_2 r) \end{bmatrix} e^{jn\phi} e^{jhz} = -\frac{1}{k_-} \boldsymbol{M_{2\lambda_2 n}}(h). \tag{11d}$$

In (10) and (11), the subscripts λ_1, λ_2, and n attached to the vector wave functions designating discrete eigenvalues and h is determined from the dispersion equations $\lambda_1^2 + h^2 = k_+^2, \lambda_2^2 + h^2 = k_-^2$. $J_n(\lambda_1 r)$ and $J_n(\lambda_2 r)$ are Bessel functions of the first kind and order n. The coefficients $A_{\lambda_1 n}$ and $B_{\lambda_2 n}$ and the eigenvalues λ_1 and λ_2 are determined by matching the boundary condition of the electric field on the surface of the cylindrical chirowaveguide [11]. That is

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{n} \times (t\boldsymbol{Q_1} + \boldsymbol{Q_2}) = \boldsymbol{0}. \tag{12}$$

When (12) is satisfied, $\boldsymbol{E}$ and $\boldsymbol{H}$ can be expressed as linear combinations of $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$ as in (5). It was found that these modes are mutually orthogonal (eq. (21) in [12]) for a time-harmonic electromagnetic field and therefore we can write

$$\boldsymbol{E}(\boldsymbol{R}) = \sum_{\lambda_1 \lambda_2 n} \Gamma_{\lambda_1 \lambda_2 n}(\pm h)[t\boldsymbol{Q_{1\lambda_1 n}}(\pm h) + \boldsymbol{Q_{2\lambda_2 n}}(\pm h)] \quad z \gtrless z' \tag{13a}$$

$$\boldsymbol{H}(\boldsymbol{R}) = \sum_{\lambda_1 \lambda_2 n} \Gamma_{\lambda_1 \lambda_2 n}(\pm h)[\boldsymbol{Q_{1\lambda_1 n}}(\pm h) - \frac{1}{t}\,\boldsymbol{Q_{2\lambda_2 n}}(\pm h)] \quad z \gtrless z' \tag{13b}$$

where $\Gamma_{\lambda_1 \lambda_2 n}(\pm h)$ are expansion coefficients that are readily determined by using the method given in [13]. In (13a) and (13b) the upper lines are for modes propagating in the positive z direction while the lower lines are for those propagating in the negative z direction. Note that (13a) is valid only outside the source point. The solenoidal magnetic dyadic Green's function $\overline{\overline{\boldsymbol{G}}}_{\boldsymbol{m}}$,however, can still be found from (13b) by using the following relationship:

$$\boldsymbol{H}(\boldsymbol{R} = \iiint_V \overline{\overline{\boldsymbol{G}}}_{\boldsymbol{m}}(\boldsymbol{R}, \boldsymbol{R}') \cdot \boldsymbol{J}(\boldsymbol{R}')\, dv' \tag{14}$$

where V is the volume containing the current source, $\boldsymbol{J}(\boldsymbol{R}')$. The result is

$$\begin{aligned} \overline{\overline{\boldsymbol{G}}}_{\boldsymbol{m}}(\boldsymbol{R}, \boldsymbol{R}') = \sum_{\lambda_1 \lambda_2 n} & \frac{1}{I_{\lambda_1 \lambda_2 n}(\mp h)(1 - k^2\beta^2)} \\ & \cdot \Bigg[t\left(1 + j\,\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right)\boldsymbol{Q_{1\lambda_1 n}}(\pm h) \\ & \cdot \boldsymbol{Q'_{1\lambda_1(-n)}}(\mp h) + \left(1 - j\,\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right) \\ & \cdot \boldsymbol{Q_{1\lambda_1 n}}(\pm h)\boldsymbol{Q'_{2\lambda_2(-n)}}(\mp h) \\ & - \left(1 + j\,\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right)\boldsymbol{Q_{2\lambda_2 n}}(\pm h) \\ & \cdot \boldsymbol{Q'_{1\lambda_1(-n)}}(\mp h) - \frac{1}{t}\left(1 - j\,\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right) \\ & \cdot \boldsymbol{Q_{2\lambda_2 n}}(\pm h)\boldsymbol{Q'_{2\lambda_2(-n)}}(\mp h)\Bigg] \quad z \gtrless z' \end{aligned} \tag{15}$$

where the primed functions are defined with respect to the source coordinates and $I_{\lambda_1 \lambda_2 n}(\mp h)$ are normalization constants given by

$$I_{\lambda_1\lambda_2 n}(\mp h) = \pm(-1)^n 8\pi j \Bigg\{ tA^2_{\lambda_1 n} \Bigg\{ \left(1+\frac{h^2}{k_+^2}\right)\frac{n}{2} \cdot [J_n^2(\lambda_1 a) - \delta_{n0}] \mp \frac{a^2 h}{4k_+}\Bigg\{ \left[\lambda_1 - \frac{(n-1)^2}{a^2}\right] \cdot J^2_{n-1}(\lambda_1 a) + \left[\frac{\partial J_{n-1}(\lambda_1 r)}{\partial r}\bigg|_{r=a}\right]^2 + \left[\lambda_1 - \frac{(n+1)^2}{a^2}\right] J^2_{n+1}(\lambda_1 a) + \left[\frac{\partial J_{n+1}(\lambda_1 r)}{\partial r}\bigg|_{r=a}\right]^2\Bigg\}\Bigg\} - \frac{B^2_{\lambda_2 n}}{t} \cdot \Bigg\{\left(1+\frac{h^2}{k_-^2}\right)\frac{n}{2}\,[J_n^2(\lambda_2 a) - \delta_{n0}] \pm \frac{a^2 h}{4k_-} \cdot \Bigg\{\left[\lambda_2 - \frac{(n-1)^2}{a^2}\right] J^2_{n-1}(\lambda_2 a) + \left[\frac{\partial J_{n-1}(\lambda_2 r)}{\partial r}\bigg|_{r=a}\right]^2 + \left[\lambda_2 - \frac{(n+1)^2}{a^2}\right] \cdot J^2_{n+1}(\lambda_2 a) + \left[\frac{\partial J_{n+1}(\lambda_2 r)}{\partial r}\bigg|_{r=1}\right]^2\Bigg\}\Bigg\}\Bigg\} \tag{16}$$

in which

$$\delta_{n0} = \begin{cases} 1, & \text{when } n = 0 \\ 0, & \text{when } n \neq 0. \end{cases}$$

Finally $\overline{\overline{G}}_e$ is derived from $\overline{\overline{G}}_m$ as in [14] by treating $\overline{\overline{G}}_m$ as a generalized function. The singular term in $\overline{\overline{G}}_e$ results from a differential operation on the discontinuous $\overline{\overline{G}}_m$, i.e.,

$$\overline{\overline{G}}_e(\boldsymbol{R},\boldsymbol{R}') = \frac{1-k^2\beta^2}{\gamma^2}\nabla\times\overline{\overline{G}}_m(\boldsymbol{R},\boldsymbol{R}') - \beta(1-k^2\beta^2)\overline{\overline{G}}_m(\boldsymbol{R},\boldsymbol{R}') - \frac{1}{\gamma^2}\overline{\overline{I}}\delta(\boldsymbol{R}-\boldsymbol{R}')$$
$$= -\frac{1}{\gamma^2}\boldsymbol{zz}\delta(\boldsymbol{R},\boldsymbol{R}') \cdot \sum_{\lambda_1\lambda_2 n}\frac{1}{I_{\lambda_1\lambda_2 n}(\mp h)\gamma\sqrt{1+\gamma^2\beta^2}} \cdot \Bigg[t\left(1+j\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right)\boldsymbol{Q_{1\lambda_1 n}}(\pm h) \cdot \boldsymbol{Q}'_{\boldsymbol{1}\lambda_1(-n)}(\mp h) + \left(1-j\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right) \cdot \boldsymbol{Q_{1\lambda_1 n}}(\pm h)\boldsymbol{Q}'_{\boldsymbol{2}\lambda_2(-n)}(\mp h) + \left(1+j\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right)\boldsymbol{Q_{2\lambda_2 n}}(\pm h) \cdot \boldsymbol{Q}'_{\boldsymbol{1}\lambda_1(-n)}(\mp h) + \frac{1}{t}\left(1-j\frac{k\beta}{t}\sqrt{\frac{\mu}{\epsilon}}\right) \cdot \boldsymbol{Q_{2\lambda_2 n}}(\pm h)\boldsymbol{Q}'_{\boldsymbol{2}\lambda_2(-n)}(\mp h)\Bigg] \quad z \gtrless z' \tag{17}$$

where $\boldsymbol{z}$ is a unit vector in the z direction. Note that the singular term in $\overline{\overline{G}}_e$ is the same as those appearing in the electric dyadic Green's functions for nonchiral cylindrical waveguides [14].

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